MATH 151 Exam 1 Notes Spring

1 Integration by Parts

Recall that *integration* and *differentiation* are related. How?

Each differentiation rule has a corresponding integration rule

chain rule for differentiation $\longleftrightarrow$ substitution for integration

product rule for differentiation $\longleftrightarrow$ integration by parts

Recall the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

if we let $u = f(x)$ and $v = g(x)$ then

$$\frac{d}{dx}[uv] = udv + vdu$$

integrating both sides gives us....

$$uv = \int u \; dv + \int v \; du$$

or
the formula for integration by parts:

$$\int u \; dv = uv - \int v \; du$$

or, using $f(x)$ and $g(x)$

$$\int f(x)g'(x) \; dx = f(x)g(x) - \int g(x)f'(x) \; dx$$

ex 1 find

$$\int x\ sin\ x\ \ dx$$

Since

$$\int u\ \ dv = uv - \int v\ \ du$$

we need to choose **both** a u and a dv

the idea is to choose the u such that du gets "easier"

So let $u = x$, then $du = 1$, which makes $dv = sin\ x$ and $v = -cos\ x$

note that we can choose *any* antiderivative so just choose the simplest one, that is let $C = 0$

thus

$$\int x\ sin\ x\ \ dx = x(-cos\ x) - \int (-cos\ x)\ \ dx$$
$$= -x\ cos\ x + \int (cos\ x)\ \ dx$$
$$= -x\ cos\ x + sin\ x + C$$

if we had let $u = sin\ x$ and $dv = x\ dx$, we would have gotten

$$\int x\ sin\ x\ \ dx = \frac{x^2}{2}(sin\ x) - \frac{1}{2}\int x^2\ cos\ x\ \ dx$$

which clearly is NOT easier.

ex 2 Evaluate

$$\int ln\ x\ \ dx$$

So let $u = ln\ x$, then $dv = dx$, which makes $du = 1/x\ dx$ and $v = x$

this gives us

$$\ldots \int ln\ x\ \ dx = x\ ln\ x - x + C$$

Sometimes you may need to do parts twice...

ex 3 Evaluate

$$\int x^2 \ cos \ mx \ \ dx$$

So let $u = x^2$, then $dv = cos \ mx \ dx$, which makes $du = 2x \ dx$ and $v = 1/m \ sin \ mx$

thus

$$\int x^2 \ cos \ mx \ \ dx = 1/m \ x^2 \ sin \ mx - 2/m \ \int x \ sin \ mx \ \ dx$$

Yes, this has gotten a little easier but we still need to evaluate $\int x \ sin \ mx \ \ dx$

so use parts again with $U = x$, then $dV = sin \ mx \ dx$, which makes $dU = dx$ and $V = -1/m \ cos \ mx$

therefore

$$\begin{aligned}\int x^2 \ cos \ mx \ \ dx &= 1/m \ x^2 \ sin \ mx - 2/m[-1/m \ x \ cos \ mx + 1/m^2 \ sin \ mx + C] \\ &= 1/m \ x^2 \ sin \ mx + 2/m^2 \ x \ cos \ mx - 2/m^3 \ sin \ mx + C\end{aligned}$$

This next example shows one final trick or technique for integration by parts. You will notice that neither u or dv gets any easier but occasionally you can perform parts twice and get back the original function. When this happens you simply gather all the like terms, in this case the original integral, back over to the left hand side and simplify.

Observe:

ex 4 Evaluate

$$\int e^x \ sin \ x \ \ dx$$

So let $u = e^x$, then $dv = sin \ x \ dx$, which makes $du = e^x$ and $v = -cos \ x$

so

$$(1) \quad \int e^x \ sin \ x \ \ dx = -e^x \ cos \ x + \int e^x \ cos \ x \ \ dx$$

now, just looking at $\int e^x \ cos \ x \ \ dx$

let $u = e^x$, then $dv = cos \ x \ dx$, which makes $du = e^x$ and $v = sin \ x$

so now we have

$$(2) \quad \int e^x \ cos \ x \ \ dx = e^x \ sin \ x - \int e^x \ sin \ x \ \ dx$$

and here you can see that we get back the original integral

combining (1) and (2) yields

$$\int e^x \ sin \ x \ \ dx = -e^x \ cos \ x + e^x \ sin \ x - \int e^x \ sin \ x \ \ dx$$

or

$$2 \int e^x \ sin \ x \ \ dx = e^x (sin \ x - cos \ x)$$

so it follows that

$$\int e^x \ sin \ x \ \ dx = \frac{1}{2} \ e^x \ (sin \ x - cos \ x) + C$$

You can also use parts for Definite Integrals

$$\int_a^b f(x) g'(x) \ \ dx = f(x) g(x) \Big|_a^b - \int_a^b g(x) f'(x) \ \ dx$$

2 Trigonometric Integrals

2.1 Integrals of the type

$$\int sin^m\ x\ cos^n\ x\ \ dx$$

ex 5 Evaluate

$$\int sin^3\ x\ \ dx$$

Clearly substitution will not work here

to integrate $cos\ x$ you need a $sin\ x$

to integrate $sin\ x$ you need a $cos\ x$

use the trig identities, that is,

$$cos^2 x + sin^2\ x = 1$$

thus

$$\int sin^3\ x\ \ dx = \int sin^2\ x\ sin\ x\ \ dx = \int (1 - cos^2\ x) sin\ x\ \ dx$$

then let $u = cos\ x$ and $du = -sin\ x$

so

$$\int sin^3\ x\ \ dx = \ldots = -cos\ x + \frac{1}{3} cos^3\ x + C$$

The idea is: write the integrand with
(1) One factor of sine and the rest cosine

or

(2) One factor of cosine and the rest sine

ex 6 Evaluate

$$\int sin^5\ x\ cos^2\ x\ \ dx$$

rewriting...

$$\int sin^5\ x\ cos^2\ x\ \ dx = \int (sin^2\ x)^2\ cos^2\ x\ sin\ x\ \ dx$$

then substitution yields:

$$-\int (1-u^2)^2\ u^2\ \ du = \ldots = -\frac{1}{3}cos^3\ x + \frac{2}{5}cos^5\ x - \frac{1}{7}cos^7\ x + C$$

As long as you have odd powers the preceding strategy is great but what if there are only even powers of sine and cosine?

Use the half-angle identities:

$$sin^2 x = \frac{1}{2}(1 - cos\ 2x)$$

or

$$cos^2 x = \frac{1}{2}(1 + cos\ 2x)$$

ex 7 Find

$$\int sin^4\ x\ \ dx$$

rewriting...

$$\int sin^4\ x\ \ dx = \int (sin^2\ x)^2\ \ dx$$

and substituting the identities yields:

$$\int \left(\frac{1 - cos\ 2x}{2}\right)^2\ dx = \frac{1}{4}\int (1{-}2\ cos\ 2x{+}cos^2\ 2x)\ \ dx = \ldots = \frac{1}{4}\left(\frac{3}{2}\ x - sin\ 2x + \frac{1}{8}\ sin\ 4x\right){+}C$$

So, the strategy for $\int\ sin^m\ x\ cos^n\ x\ \ dx$ is:

1. If the power of $cos\ x$ is odd, save one $cos\ x$ and make the rest $sin\ x$ and let $u = sin\ x$
2. If the power of $sin\ x$ is odd, save one $sin\ x$ and make the rest $cos\ x$ and let $u = cos\ x$
3. If both have even powers then use the half-angle identities

2.2 Integrals of the type

$$\int tan^m\ x\ sec^n\ x\ \ dx$$

Here the strategy is slightly different and you will be using the identity:

$$sec^2\ x = 1 + tan^2\ x$$

or

$$tan^2\ x = sec^2\ x - 1$$

1. If the power of $sec\ x$ is even, save a $sec^2\ x$ and make the rest $tan\ x$ and let $u = tan\ x$
2. If the power of $tan\ x$ is odd, save one $sec\ x\ tan\ x$ and make the rest $sec\ x$ and let $u = sec\ x$
3. Otherwise, be creative

ex 8 Find

$$\int_0^{\pi/4} sec^4\ \theta\ tan^4\ \theta\ \ d\theta$$

rewriting...

$$\int_0^{\pi/4} sec^4\ \theta\ tan^4\ \theta\ \ d\theta = \int_0^{\pi/4} (tan^2\ \theta + 1)(tan^4\ \theta) sec^2\ \theta\ \ d\theta$$

and using the substitution $u = tan\ \theta$ and $du = sec^2\ \theta\ d\theta$ yields:

$$\int_0^{\pi/4} (u^2 + 1)(u^4)\ \ du = \ldots = \frac{12}{35}$$

3 Trigonometric Substitution

Now we will use a slightly different kind of substitution called *inverse substitution*

This method enables us to deal with integrals of the type

$$\int \sqrt{a^2 - x^2}\ dx$$

There are three kinds:

Expression	**Substitution and Restriction**	**Identity**
$\sqrt{a^2 - x^2}$	$x = a\ sin\ \theta\ -\pi/2 \leq \theta \leq \pi/2$	$1 - sin^2\ \theta = cos^2\ \theta$
$\sqrt{a^2 + x^2}$	$x = a\ tan\ \theta\ -\pi/2 < \theta < \pi/2$	$1 + tan^2\ \theta = sec^2\ \theta$
$\sqrt{x^2 - a^2}$	$x = a\ sec\ \theta\ 0 \leq \theta < \pi/2$	$sec^2\ \theta - 1 = tan^2\ \theta$

* Think of this as using trig and the triangle as a substitution *

ex 9 Evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2}\ dx$$

If we let $x = 3\ sin\ \theta$ then we have that $dx = 3\ cos\ \theta\ d\theta$ and

$\sqrt{9 - x^2} = \sqrt{9 - 9\ sin^2\ \theta} = \sqrt{9\ cos^2\ \theta} = 3\ \ |\ cos\ \theta\ |\ \ = 3\ cos\ \theta$ from the restriction.

Why can we substitute this way?

Think of a right triangle whose opposite side is x, adjacent is $\sqrt{9 - x^2}$ and whose hypotenuse is 3...

So, inverse substitution yields:

$$\int \frac{\sqrt{9 - x^2}}{x^2}\ dx = \int \frac{3\ cos\ \theta}{9\ sin^2\ \theta}\ 3\ cos\ \theta\ \ d\theta$$

$$= \int cot^2\ \theta\ \ d\theta = \int (csc^2\ \theta - 1)\ \ d\theta$$

$$= -cot\ \theta - \theta + C = -\frac{\sqrt{9 - x^2}}{x} - sin^{-1}\ \left(\frac{x}{3}\right) + C$$

Note that we must return to the original x. We accomplish this by revisiting our triangle that was created for the substitution.

ex 10 Evaluate

$$\int_0^2 x^3 \sqrt{x^2+4}\ dx$$

If we let $x = 2\ tan\ \theta$ then we have that $dx = 2\ sec^2\ \theta\ d\theta$

also note that we can change the bounds at this point. When $x = 0$, $\theta = 0$ and when $x = 2$, $tan\ \theta = 1$ so $\theta = \pi/4$

Here we want to think of a right triangle whose opposite side is x, adjacent is 2 and whose hypotenuse is $\sqrt{x^2+4}$

So, inverse substitution yields:

$$\int_0^2 x^3 \sqrt{x^2+4}\ dx = \int_0^{\pi/4} 2^3\ tan^3\ \theta\ \sqrt{4\ tan^2\ \theta + 4}\ 2\ sec^2\ \theta\ d\theta$$

$$= 2^5 \int_0^{\pi/4} tan^3\ \theta\ sec\ \theta\ sec^2\ \theta\ d\theta = 2^5 \int_0^{\pi/4} (sec^2\ \theta - 1)\ sec^2\ \theta\ sec\ \theta\ tan\ \theta\ d\theta$$

and if we let $u = sec\ \theta$ and $du = sec\ \theta\ tan\ \theta\ d\theta$ then we have:

$$2^5 \int_0^{\pi/4} (u^4 - u^2)\ du = \quad \ldots \quad = \frac{64}{15}\left(\sqrt{2}+1\right)$$

4 Integration of Rational Functions using Partial Fractions

It should be routine to recall how to combine

$$\frac{2}{x-1} - \frac{1}{x+2} = \quad \dots \quad = \frac{x+5}{x^2+x-2}$$

Partial fraction decomposition reverses this process...

Why bother with this? Then it will be possible to integrate things like

$$\int \frac{x+5}{x^2+x-2}\ dx$$

$$= \quad \dots \quad = \int \frac{2}{x-1}\ dx - \int \frac{1}{x+2}\ dx$$

In other words we can now integrate *Rational Functions*
Before we begin, the rational function in question must be *proper*, that is, the degree of the numerator must be less than the degree of the denominator.

If not, then you need to divide. Sometimes this is all that is necessary.

ex 11 Evaluate

$$\int \frac{x^3+x}{x-1}\ dx$$

This is an improper rational function so division gives us...

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

thus

$$\int \frac{x^3+x}{x-1}\ dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)\ dx$$
$$= \quad \dots \quad = \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ ln\ \mid x-1\mid +C$$

The focus will be on the denominator of the rational function as it ***must*** be factored as much as possible.

Fortunately there is a theorem that says ANY polynomial can be factored as a product of linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c). As you may have guessed this will yield four cases...

4.1 CASE I: Distinct Linear Factors

So, by the previous theorem there must exist constants

$$A_1, A_2, \ldots, A_k$$

such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_k}{a_kx + b_k}$$

ex 12 write the partial fraction decomposition of

$$\frac{1}{x^2 - 5x + 6}$$

so,

$$Q(x) = x^2 - 5x + 6 = (x - 3)(x - 2)$$

and we have two distinct linear factors

thus

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

multiply both sides of the equation by $Q(x)$ and get

$$1 = A(x - 2) + B(x - 3)$$

Now we have 2 options:

1. substitute values for x *or*
2. equate the coefficients

1. Since

$$\begin{aligned} &1 = A(x - 2) + B(x - 3) \\ &x = 3 \Longrightarrow A = 1 \\ &x = 2 \Longrightarrow B = -1 \end{aligned}$$

or

2. Since

$$
\begin{aligned}
&1 = A(x-2) + B(x-3)\\
&1 = (A+B)x - 2A - 3B\\
&A + B = 0 \Longrightarrow A = -B\\
&also - 2A - 3B = 1 \Longrightarrow A = 1 \ and \ B = -1
\end{aligned}
$$

It depends on the problem as to which method is easier. The first method usually works better on linear factors and the second usually works better with quadratic factors.

ex 13 Evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \ dx$$

so

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

since we have 3 distinct linear factors. Expanding gives us

$$x^2 + 2x - 1 = (2A + B + 2c)x^2 + (3A + 2B - C)x - 2A$$

thus

$$
\begin{aligned}
2A + B + 2C = 1 \quad &\Longrightarrow \quad A = \frac{1}{2}\\
3A + 2B - C = 2 \quad &\Longrightarrow \quad B = \frac{1}{5}\\
-2A = -1 \quad &\Longrightarrow \quad C = -\frac{1}{10}
\end{aligned}
$$

therefore

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \ dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right] \ dx$$

$$= \quad \ldots \quad = \quad \frac{1}{2} \ ln \ \mid x \mid + \frac{1}{10} \ ln \ \mid 2x-1 \mid - \frac{1}{10} \ ln \ \mid x+2 \mid + C$$

4.2 CASE II: Repeated Linear Factors

Suppose that a linear factor, $(a_1x + b_1)$ is repeated r times, that is, $(a_1x + b_1)^r$ is in the factorization of $Q(x)$.

Then instead of

$$\frac{A_1}{a_1x + b_1}$$

you will need

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \ldots + \frac{A_r}{(a_1x + b_1)^r}$$

ex 14 Evaluate

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\ dx$$

so

$$x^3 + 2x^2 + x = x(x+1)^2$$

thus

$$\frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

and

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx$$

$$If\ x = 0 \quad \Longrightarrow \quad A = 6$$
$$If\ x = -1 \quad \Longrightarrow \quad C = 9$$

to find B use any other value of x along with $A = 6$ and $C = 9$

using $x = 1 \quad \Longrightarrow \quad 31 = 4A + 2B + C \quad \Longrightarrow \quad B = -1$

therefore

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\ dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right]\ dx$$

$$= \quad \ldots \quad = \quad ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$$

NOTE:

you must make as many substitutions as there are unknowns with this method. We used $x = 0, 1$ and 2.

4.3 CASE III: Distinct Linear and Irreducible Quadratic Factors

Now you will have a term of the type:

$$\frac{Ax+B}{ax^2+bx+c}$$

Also, you will need the following formula:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\, tan^{-1} \left(\frac{x}{a}\right) + C$$

ex 15 Evaluate

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\ dx$$

so

$$(x^2-x)(x^2+4) = x(x-1)(x^2+4)$$

thus

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

and

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

$$\begin{aligned} If\ x=0 \quad &\Longrightarrow\ A=2 \\ If\ x=1 \quad &\Longrightarrow\ B=-2 \\ If\ x=-1, A=2, B=-2 \quad &\Longrightarrow\ C=2 \\ If\ x=2, A=2, B=-2 \quad &\Longrightarrow\ D=4 \end{aligned}$$

therefore

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\ dx = \int \left[\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right]\ dx$$

$$= \quad \ldots \quad = 2\ ln\ \mid x \mid -2\ ln\ \mid x-1 \mid + ln\ (x^2+4) + 2\ tan^{-1}\left(\frac{x}{2}\right) + C$$

4.4 CASE IV: Repeated Linear and Irreducible Quadratic Factors

If $Q(x)$ has a factor $(ax^2+bx+c)^r$ then the partial fraction decomposition must have:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Case IV will require you to equate coefficients to solve for the constants

ex 16 Evaluate

$$\int \frac{8x^3+13x}{(x^2+2)^2}\ dx$$

so

$$\frac{8x^3+13x}{(x^2+2)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$$

and

$$8x^3+13x=(Ax+B)(x^2+2)+Cx+D$$
$$\Longrightarrow\quad 8x^3+13x=Ax^3+Bx^2+(2A+C)x+(2B+D)$$

$$\Longrightarrow\quad A=8$$
$$\Longrightarrow\quad B=0$$
$$2A+C=13\qquad\Longrightarrow\quad C=-3$$
$$2B+D=0\qquad\Longrightarrow\quad D=0$$

therefore

$$\int \frac{8x^3+13x}{(x^2+2)^2}\ dx=\int\left[\frac{8x}{x^2+2}-\frac{3x}{(x^2+2)^2}\right]\ dx$$

$$=\quad\ldots\quad=\ 4\ ln\ (x^2+2)+\frac{3}{2(x^2+2)}+C$$

There are times when some non-rational functions can be changed into rational functions by a clever substitution.

ex 17 Evaluate

$$\int \frac{\sqrt{x+4}}{x}\ dx$$

If we let $u = \sqrt{x+4}$, then $u^2 = x+4$, $x = u^2 - 4$ and $dx = 2udu$
thus

$$\int \frac{\sqrt{x+4}}{x}\ dx = \int \frac{u}{u^2-4}\ 2u\ du =$$

after dividing the rational function

$$= 2\ \int \left[1 + \frac{4}{u^2-4}\right]\ du$$

and then doing partial fractions on $u^2 - 4 = (u+2)(u-2)$ we get...

$$\ldots\quad 2u + 2(ln\ \ \mid u-2 \mid -ln\ \ \mid u+2 \mid) + C$$
$$= 2\sqrt{x+4} + 2(ln\ \ \mid \sqrt{x+4} - 2 \mid -ln\ \ \mid \sqrt{x+4} + 2 \mid) + C$$

it is not always necessary to use partial fractions on all rational functions

ex 18 Evaluate

$$\int \frac{x^2+1}{x^3+3x-4}\ dx$$

$$= \frac{1}{3}\ \int \frac{3x^2+3}{x^3+3x-4}\ dx\ \ =\ \ \int \frac{1}{u}\ du$$
$$= \frac{1}{3}\ ln\ \ \mid x^3+3x-4 \mid + C$$

ex 19 Find

$$\int \frac{x^2-x-2}{x^3-2x-4}\ dx$$

$$= \int \frac{(x+1)(x-2)}{(x-2)(x^2+2x+2)}\ dx\ \ =\ \ \int \frac{x+1}{x^2+2x+1}\ dx$$
$$= \frac{1}{2}\ ln\ \ \mid x^2+2x+2 \mid + C$$

5 Integration Strategies

If you do not immediately see how to integrate, try this 4 step process:

1. Simplify the integrand

$$\int (sin\ x + cos\ x)^2\ dx \quad = \quad \ldots \quad = \int (1 + 2\ sin\ x\ cos\ x)\ dx \quad = \quad \ldots$$

2. Look for the substitution

$$\int \frac{x}{x^2 - 1}\ dx$$

   Here partial fractions is **NOT** necessary

3. Try to classify the integrand:

   (a) trig functions $\longrightarrow$ use trig substitution
   (b) rational functions $\longrightarrow$ use partial fractions
   (c) (polynomial)(transcendental) $\longrightarrow$ use integration by parts
   (d) radicals in the integrand $\longrightarrow$ use trig substitution or clever substitution ($u = \sqrt{stuff}$)

4. If all else fails just realize there are really only two methods: substitution and parts

   (a) try a clever substitution or
   (b) try integration by parts or
   (c) try manipulation using identities, rationalization, etc...

ex 20 Find

$$\int \frac{dx}{1 - cos\ x}$$

$$= \int \left(\frac{1}{1 - cos\ x}\right)\left(\frac{1 + cos\ x}{1 + cos\ x}\right)\ dx \quad = \quad \int \frac{1 + cos\ x}{sin^2\ x}\ dx$$

$$= \int \left(csc^2\ x + \frac{cos\ x}{sin^2\ x}\right)\ dx = \ldots$$

6 Integration using tables

Sometimes integrals are difficult to evaluate by hand

We can use a computer system OR we can use tables of integrals like those listed at **www.integral-table.com**

Sometimes you may have to rearrange the integrand so that it corresponds with the table entry

Not everything has an antiderivative that is an elementary function (what are those?)

ex 21 Find

$$\int \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \; dx$$

This is a very important integral in statistics, do you know what this integral represents?

7 Improper Integrals

The Fundamental Theorem of Calculus assumes two things, what are they?

1. The definite integral must have a finite interval, that is on $[a, b]$ and
2. The integrand, $f(x)$ must be continuous

Violating either of these results in what is called an *Improper Integral*
There are 2 types:

7.1 TYPE I: Infinite Intervals

ex 22 Consider the following where $b > 1$

$$\int_1^b \frac{1}{x^2}\ dx$$

$$= -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

note that no matter how large b is, the area under the curve is always less than 1

that is

$$\textit{as } b \longrightarrow \infty \qquad \int_1^\infty \frac{1}{x^2}\ dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\ dx = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$

Definition of Type I

1. If $\displaystyle\int_a^t f(x)\,dx$ exists $\forall\ t \geq a$, then

   $$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

   provided the limit exists ***as a finite number***

2. If $\displaystyle\int_t^b f(x)\,dx$ exists $\forall\ t \leq b$, then

   $$\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

   provided the limit exists ***as a finite number***

The *Improper Integrals*

$$\int_a^\infty f(x)\ dx \ \ and \ \ \int_{-\infty}^b f(x)\ dx$$

are called **convergent**, or what I will write as **C** if the corresponding limit exists and

divergent, or **D** if the limit DNE

3. If both

$$\int_a^\infty f(x)\ dx \ \ and \ \ \int_{-\infty}^a f(x)\ dx$$

are convergent then

$$\int_{-\infty}^\infty f(x)\ dx \ = \ \int_{-\infty}^a f(x)\ dx \ + \ \int_a^\infty f(x)\ dx$$

Any real number a can be used

ex 23 Find

$$\int_1^\infty \frac{1}{x}\ dx$$

$$= \ \ldots \ = \lim_{t\to\infty}\ ln\ \mid t \mid -ln\ 1 = ln\ \infty = \infty$$

thus this integral **diverges**

ex 24 Find

$$\int_0^\infty e^{-x}\ dx$$

$$= \ \ldots \ = \lim_{t\to\infty}\ (-e^{-t}+1) = 1$$

thus this integral **converges** and is 1

ex 25 Find

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\ dx$$

$$= \ \ldots \ = \lim_{t\to-\infty}\ (tan^{-1}\ e^x)\Big|_b^0 + \lim_{t\to\infty}(tan^{-1}\ e^x)\Big|_0^t = \ldots = \frac{\pi}{2}$$

Note that:

$$\int_1^\infty \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{p-1}, & if\ p>1\ (thus\ \mathbf{C}) \\ \mathbf{D}, & if\ p \le 1 \end{cases}$$

7.2 TYPE II: Discontinuous Integrands

Definition of Type II

1. If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

provided the limit exists ***as a finite number***

2. If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

provided the limit exists ***as a finite number***

3. If f has a discontinuity at c, where $a < c < b$, then

$$\int_a^b f(x)\ dx \ = \ (1) \int_a^c f(x)\ dx \ + \ (2) \int_c^b f(x)\ dx$$

provided **both** (1) and (2) are convergent

ex 26 Find

$$\int_{-1}^2 \frac{1}{x^3}\ dx$$

- Note that we have a discontinuity at $x = 0$

$$\int_{-1}^2 \frac{1}{x^3}\ dx \ = \ \int_{-1}^0 \frac{1}{x^3}\ dx \ + \ \int_0^2 \frac{1}{x^3}\ dx$$

Now

$$\int_0^2 \frac{1}{x^3}\ dx \ = \ \lim_{t \to 0^+} \int_t^2 \frac{1}{x^3}\ dx \ = \ \lim_{t \to 0^+} \left(-\frac{1}{2x^2} \right) \Bigg|_t^2 = \ldots = \infty$$

Thus the integral **diverges**

It is important to realize that:

$$\int_{-1}^2 \frac{1}{x^3}\ dx \ \neq \ -\frac{1}{2x^2}\Bigg|_{-1}^2 = \frac{3}{8}$$

• Sometimes it is impossible to find the exact value of the integral but you would still like to know if it converges or diverges.

Comparison Theorem

Suppose that f and g are continuous and $f \geq g \geq 0$ for $x \geq a$, then:

1. If $\int_a^\infty f$ is convergent $\implies$ $\int_a^\infty g$ is convergent
2. If $\int_a^\infty g$ is divergent $\implies$ $\int_a^\infty f$ is divergent

• This is an important concept that we will return to later •

ex 27 Is the following integral **C** or **D**, that is, convergent or divergent?

$$\int_1^\infty \frac{1+e^{-x}}{x}\ dx$$

Here we will need the Comparison Theorem since the integrand has no antiderivative

Since

$$\frac{1+e^{-x}}{x} > \frac{1}{x} \implies \mathbf{D}$$