MATH 130 Exam 1 Notes

1 Functions

A *function* is a rule that assigns to each element, x, in a set A **exactly** one element, called $f(x)$, in a set B.

sometimes it is useful to think of a function as a machine, x goes into the machine and $f(x)$ comes out.

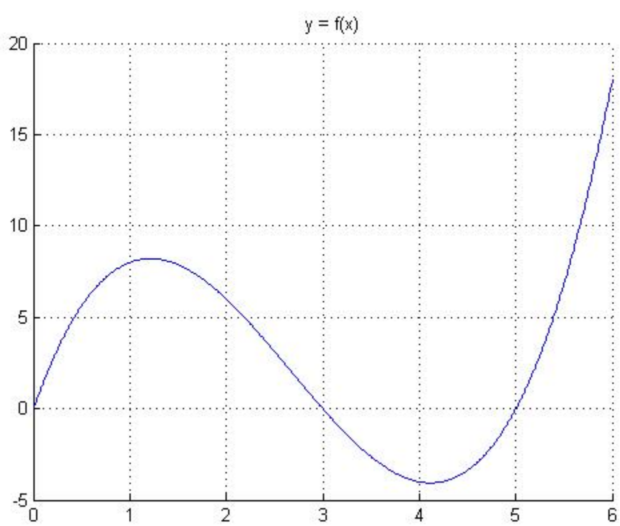
$$x \longrightarrow \boxed{\text{function}} \longrightarrow f(x)$$

The set A is called the *domain.* it corresponds to the x-axis.

The *range* of $f(x)$ is the set of all possible values of $f(x)$ as x varies through the domain.

Another way to think about it is that the domain is all of the possible inputs while the range is all of the possible outputs.

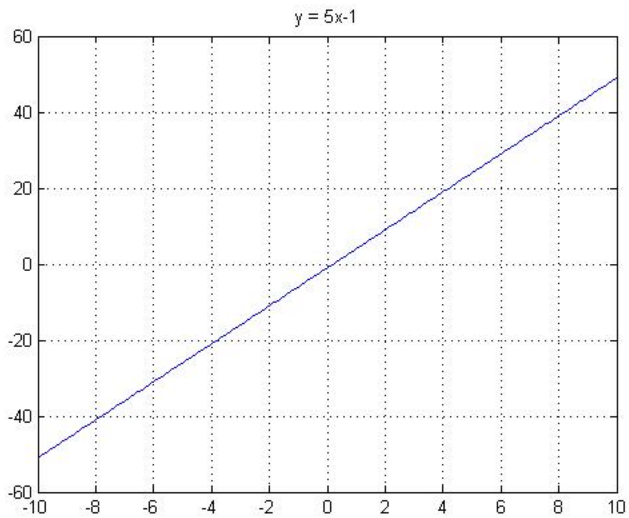
ex 1 Find the domain and range of the following function

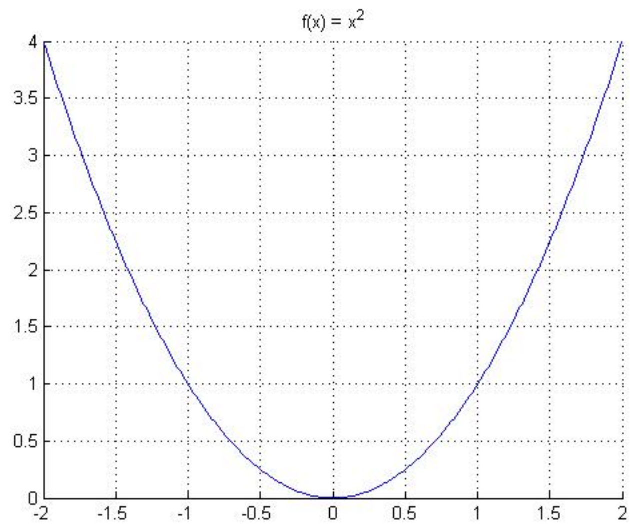


What values were covered on the x axis? Domain is $[0, 6]$
What values were covered on the y axis? Range is $[-4, 18]$

ex 2 sketch, find the domain and range of:

1. $y = 5x - 1$
2. $y = x^2$





domain=? range=?

domain=? range=?

ex 3 If $f(x) = 3x^2 - x + 4$ then find the following:

$f(1)$
$f(a)$
$f(a+h)$

$$f(1) = 3(1)^2 - (1) + 4 = 6$$

$$f(a) = 3(a)^2 - (a) + 4 = 3a^2 - a + 4$$

$$f(a+h) = 3(a+h)^2 - (a+h) + 4 = 3(a^2 + 2ah + h^2) - a - h + 4 = 3a^2 + 6ah + 3h^2 - a - h + 4$$

When finding domains, what possible problems might we have?

1. division by zero
2. negative numbers under the radical

ex 4 Find the domain of $f(x) = \sqrt{x+2}$

$$x + 2 \geq 0 \quad \iff \quad x \geq -2 \quad or \quad [-2, \infty)$$

ex 5 Find the domain of $f(x) = \sqrt{16 - x^2}$

$$16 - x^2 \geq 0 \quad \iff \quad 16 \geq x^2 \quad \iff \quad 4 \geq\mid x \mid \quad \iff \quad [-4, 4]$$

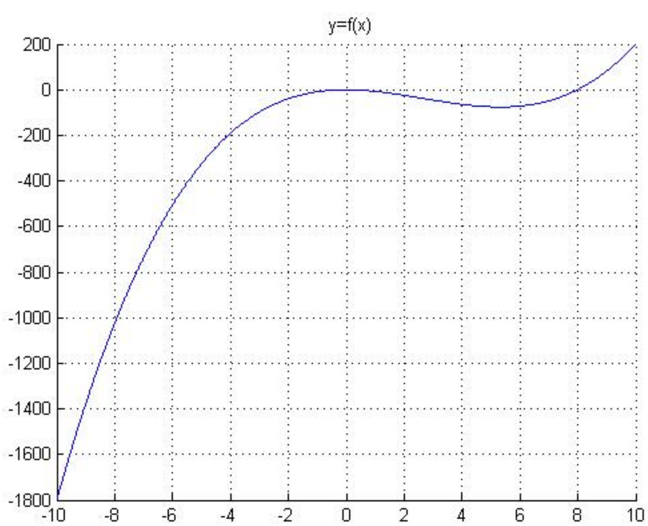
ex 6 Find the domain of:

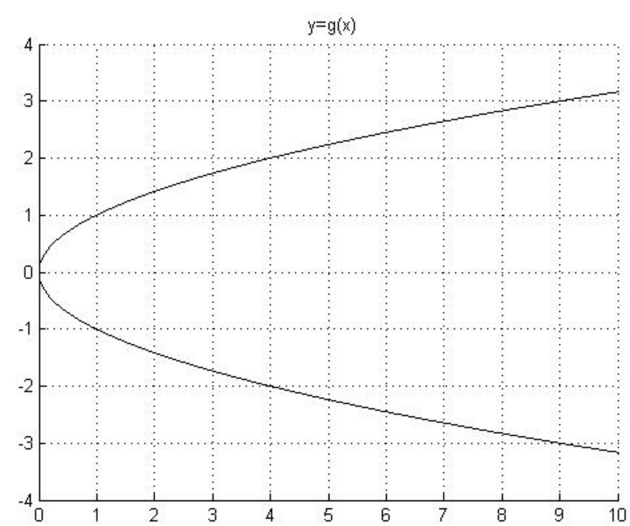
$$f(x) = \frac{1}{x^2 - x}$$

$$x^2 - x = 0 \quad \iff \quad x(x-1) = 0 \quad \implies \quad x \neq 0 \quad or \quad x \neq 1$$
$$or \quad (-\infty, 0) \ \cup \ (0, 1) \ \cup \ (1, \infty)$$

A curve in the xy plane is the graph of a function of x if and only if **no** vertical line intersects the graph more than once. Why? *(think about the definition)*

ex 7 Are the following graphs functions?





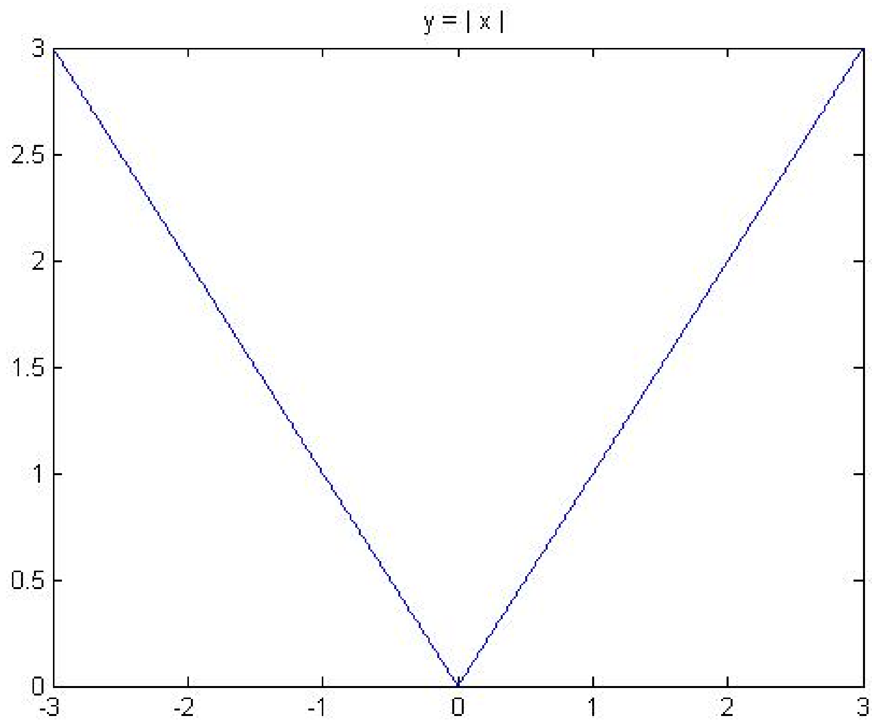
1.1 Piecewise Functions

piecewise functions are functions that are defined in parts or pieces, that is, they have different formulas for different parts of their domains.

Recall that:

$$|\ a\ | = \begin{cases} a & if \ \ a \geq 0 \\ -a & if \ \ a < 0 \end{cases}$$

ex 8 sketch $y = |\ x\ |$



2 Combining Functions

Let f and g be functions with domains A and B respectively. Then:

$(f+g)(x) = f(x) + g(x)$ where the domain is $A \cap B$
$(f-g)(x) = f(x) - g(x)$ where the domain is $A \cap B$
$(fg)(x) = f(x)g(x)$ where the domain is $A \cap B$
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ where the domain is $A \cap B$ *provided that* $g(x) \neq 0$

In other words you can add, subtract, multiply and divide functions as well

ex 9 given $f(x) = \sqrt{x}$ *and* $g(x) = \sqrt{16 - x^2}$

find:

$f + g$

fg

$\frac{f}{g}$

AND their domains

Try these on your own but we will cover them in class

2.1 Composition of Functions

Given two functions, f and g, the *composite function*, $f \circ g$, is defined as:

$$(f \circ g)(x) = f(g(x))$$

where $f \circ g$ is defined whenever both $g(x)$ and $f(g(x))$ are defined

you can think of this as a two-staged machine. That is, the input x goes into g and out comes $g(x)$ which then becomes the new input for $f(x)$.

$$x \longrightarrow \boxed{\text{g}} \longrightarrow g(x) \longrightarrow \boxed{\text{f}} \longrightarrow f(g(x))$$

ex 10 given $f(x) = x^3$ *and* $g(x) = x^2 + 1$, find the following:

$f \circ g \quad \Longrightarrow \quad f(g(x)) = f(x^2+1) = (x^2+1)^3$
$g \circ f \quad \Longrightarrow \quad g(f(x)) = g(x^3) = (x^3)^2 + 1 = x^6 + 1$
$f \circ f \quad \Longrightarrow \quad f(f(x)) = f(x^3) = (x^3)^3 = x^9$ *note that* $f \circ g \neq g \circ f$

ex 11 given that $F(x) = \sqrt{x^2 - 3}$, find f, g and h such that $F = f \circ g \circ h$ or $F = f(g(h(x)))$

Keep in mind your order of operations. To come up with the individual functions, just think if I were an input, what happens first, second and third?

first you square it.......$h(x) = x^2$
second you subtract three.......$g(x) = x - 3$
finally you take the square root of that result.......$f(x) = \sqrt{x}$

then check...

3 Types of Functions

3.1 Linear Functions

these are the graphs of lines
the main feature of a linear function is that it grows at a .... constant rate
this constant rate also goes by what name? the slope

3.2 Polynomial Functions

A function is *polynomial* if it is of the following general form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer
$a_0, a_1, \ldots$ are constants called coefficients with $a_n \neq 0$ being called the leading coefficient
and the degree of the polynomial is n

a polynomial of degree 1 is called a $\ldots$ linear function
a polynomial of degree 2 is called a $\ldots$ quadratic function
a polynomial of degree 3 is called a $\ldots$ cubic function, etc...

polynomials are useful at modeling various things because they are so easy to work with

3.3 Rational Functions

these are ratios of two polynomials

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

the domain of rational functions is all x such that $Q(x) \neq 0$

ex 12 What is the domain of f?

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 9}$$

We need $x^2 - 9 \;\neq\; 0 \quad \Longrightarrow \quad x \;\neq\; \pm 3$

3.4 Power Functions

these are functions that have the general form:

$$y = f(x) = x^a$$

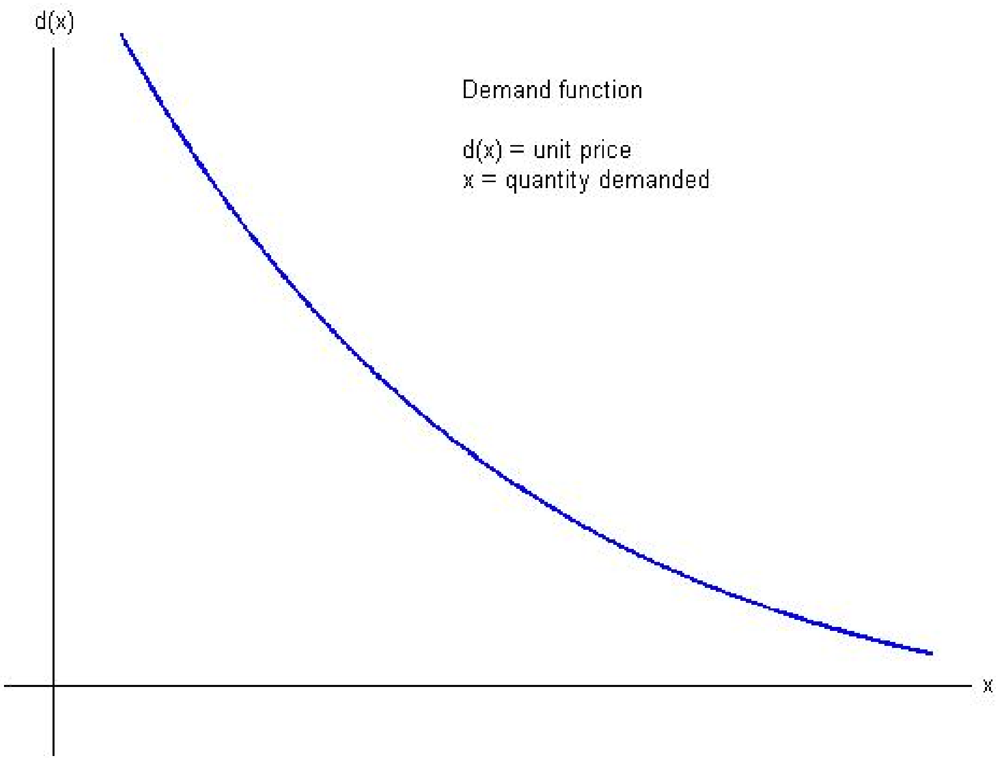
with a being a constant

3.5 Economic Models

A *demand equation* is the relationship between unit price and quantity demanded.

A *demand curve* is the graph of a demand equation.

The *demand function* looks as follows:

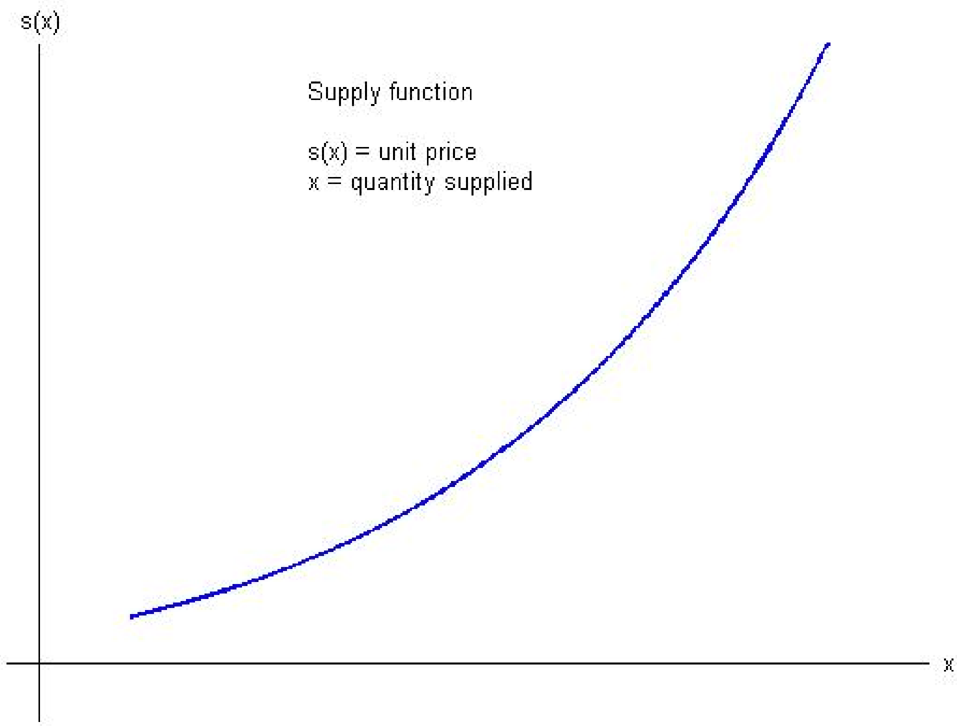


Why do you think the graph resembles this shape?

A *supply equation* is the relationship between unit price and quantity supplied.

A *supply curve* is the graph of a supply equation.

The *supply function* looks as follows:



Why do you think the graph resembles this shape?

The *market equilibrium* is when the quantity produced is equal to the quantity demanded.

The *equilibrium quantity* is the quantity produced at the market equilibrium.

The *equilibrium price* is the corresponding price at the market equilibrium.

ex 13 If the weekly supply and demand equations for a product are given below, find the equilibrium quantity AND price where x is measured in units of a thousand.

$$d(x) = 144 - x^2$$

$$s(x) = 48 + \frac{1}{2}x^2$$

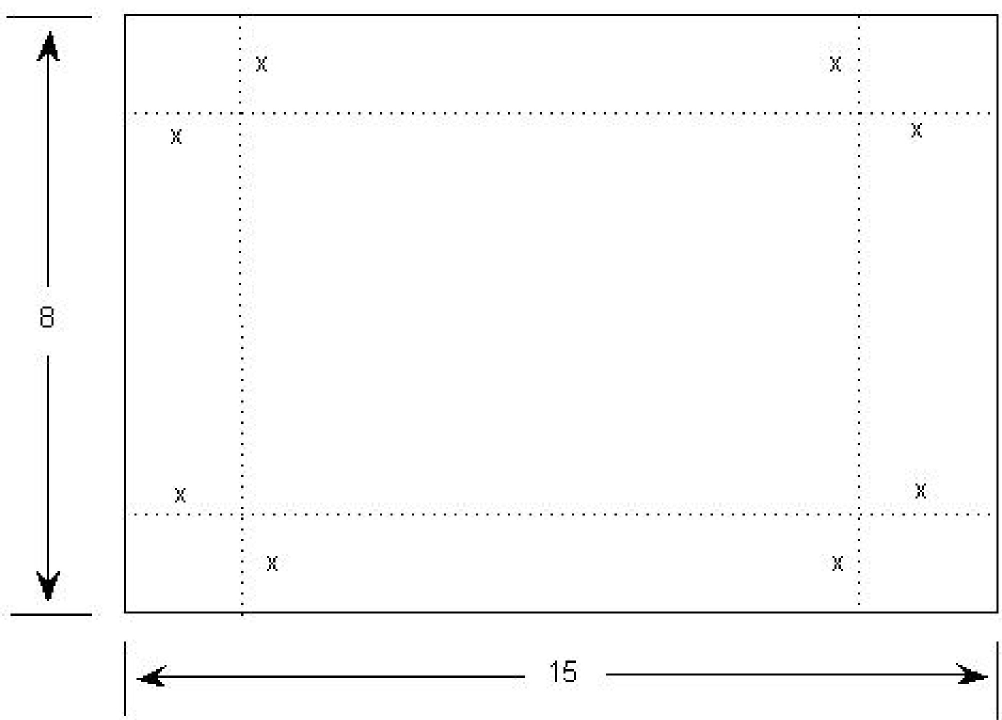
To find the equilibrium quantity:

$$\begin{aligned} 144 - x^2 &= 48 + \frac{1}{2}x^2 \\ \frac{3}{2}x^2 &= 96 \\ x^2 &= 64 \end{aligned}$$

$$x = 8 \implies \textit{the equilibrium quantity is } 8000$$

To find the equilibrium price, evaluate $d(8)$ or $s(8)$. Either will give you 80 *dollars*

ex 14 Given the following rectangular piece of cardboard, cut away x by x corners and fold it up to create an open box. Find a function that gives the volume of this box.



Since $V = (l)(w)(h) \implies V = (15 - 2x)(8 - 2x)(x)$

4 The Concept of a Limit

4.1 Velocity

Calculus is divided into two main ideas linked by a single theme

The first one we will discuss is Differentiation and the second is Integration

Both of these concepts are connected by limits

ex 15 If you take a trip in your car and you notice that after 3 hours you have travelled 180 miles, what is your average velocity?

This should be fairly straightforward

$$average\ velocity \quad = \quad \frac{distance\ traveled}{time\ elapsed} \quad = \quad \frac{180}{3} \quad = \quad 60\ \frac{mi}{hr}$$

ex 16 Suppose a ball is dropped from a tall building. Given that the distance fallen after t seconds is denoted $s(t)$ and measured in meters, we have that $s(t) = 4.9t^2$. Find the velocity of the ball after 5 seconds.

The problem here is finding the velocity at **exactly** 5 seconds, which is called the *instantaneous velocity*

Right now we have no way of calculating instantaneous velocity, or do we?

How about calculating *average velocity*?

$$average\ velocity \quad = \quad \frac{distance\ traveled}{time\ elapsed}$$

Note that instantaneous velocity is the velocity at one point and the average velocity requires two points. Let's take a look at the average velocity between 5 and 6 seconds.

$$average\ velocity \quad = \quad \frac{s(6) - s(5)}{6 - 5} \quad = \quad \frac{4.9(6)^2 - 4.9(5)^2}{6 - 5} \quad = \quad 53.9\ m/s$$

Do you think this is a good estimate? How can we get a better estimate of the velocity at 5 seconds?

Observe the following:

time interval	average velocity
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

Would you like to take a guess as to what the instantaneous velocity is when $t = 5$?

It is crucial that you understand the difference between average rate of change and instantaneous rate of change.

Average rate of change is an estimate and requires TWO points.

Instantaneous rate of change is exact and requires ONE point.

We will revisit this a little later.

4.2 Limits

ex 17 Let's take a look at the following function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Is 1 in the domain of $f(x)$?

What happens as x approaches 1 but is not equal to 1?

Take a look at the following tables:

1. As x approaches 1 from the right

x	$f(x)$
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

2. As x approaches 1 from the left

x	$f(x)$
0	1
.5	1.5
.9	1.9
.99	1.99
.999	1.999

Limits are how we formally deal with the concept of getting closer and closer

Definition

We write:

$$\lim_{x\to a}\ f(x)\ =\ L$$

We say:

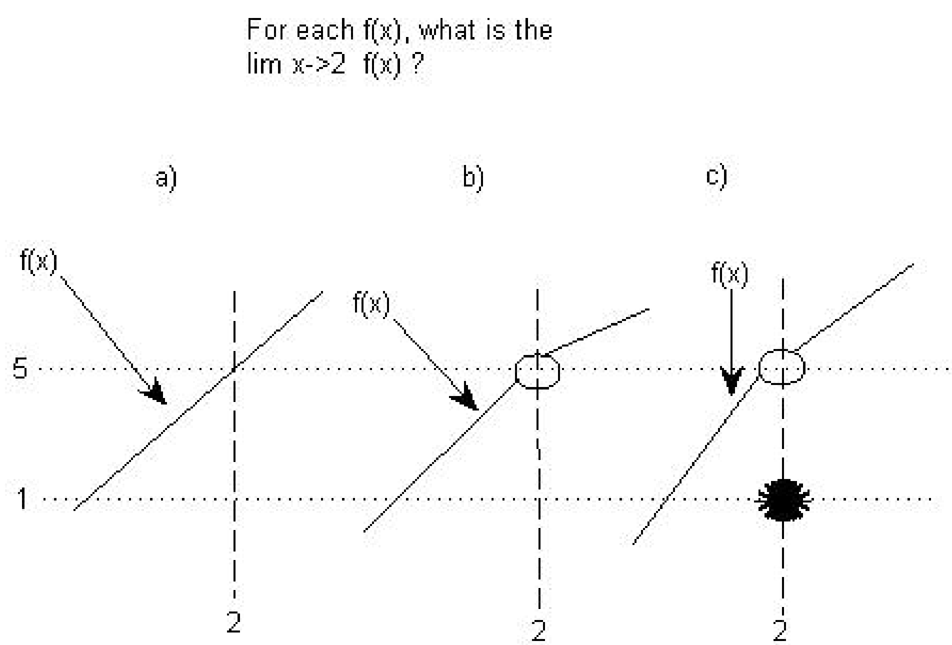
the limit of $f(x)$ as x approaches a is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently close to **BUT NOT** equal to a

In fact, $f(x)$ doesn't even need to be defined at a!

ex 18



They are all 5!

ex 19 Find

$$\lim_{x \to 0} \frac{1}{x^2}$$

Plug in some small values of x. What do you get? Do these values approach a single number?

NO, thus the limit does not exist, that is, DNE

We say

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does **NOT** mean that ∞ is a number or that the limit exists, it is simply a description as to how the limit does not exist.

Thus

We write:

$$\lim_{x \to a} f(x) = \infty$$

We say:

the limit of $f(x)$ as x approaches a is infinity

We mean:

we can make the values of $f(x)$ as large as we want by choosing values of x sufficiently close to **BUT NOT EQUAL** to a

Also

We write:

$$\lim_{x \to a} f(x) = -\infty$$

We say:

the limit of $f(x)$ as x approaches a is negative infinity

We mean:

we can make the values of $f(x)$ as large and negative as we want by choosing values of x sufficiently close to **BUT NOT EQUAL** to a

Note that f(x) need not even be defined at a

4.3 The Limit Laws

Suppose c is a constant and $\lim_{x\to a}\ f(x)$ and $\lim_{x\to a}\ g(x)$ exists. Then:

1. $\lim_{x\to a}\ [f(x)+g(x)] \ = \ \lim_{x\to a}\ f(x) \ + \ \lim_{x\to a}\ g(x)$
2. $\lim_{x\to a}\ [f(x)-g(x)] \ = \ \lim_{x\to a}\ f(x) \ - \ \lim_{x\to a}\ g(x)$
3. $\lim_{x\to a}\ [cf(x)] \ = \ c\ \lim_{x\to a}\ f(x)$
4. $\lim_{x\to a}\ [f(x)g(x)] \ = \ \lim_{x\to a}\ f(x)\ \lim_{x\to a}\ g(x)$
5. $\lim_{x\to a}\ \frac{f(x)}{g(x)} \ = \ \frac{\lim_{x\to a}\ f(x)}{\lim_{x\to a}\ g(x)}$ *provided* $\lim_{x\to a}\ g(x) \ \neq \ 0$
6. $\lim_{x\to a}\ [f(x)]^n \ = \ [\lim_{x\to a}\ f(x)]^n$ *provided* $n \ \in \ \mathbb{Z}^+$
7. $\lim_{x\to a}\ c \ = \ c$
8. $\lim_{x\to a}\ x \ = \ a$
9. $\lim_{x\to a}\ x^n \ = \ a^n$
10. $\lim_{x\to a}\ \sqrt[n]{x} \ = \ \sqrt[n]{a}$ *provided* $n \ \in \ \mathbb{Z}^+$
11. $\lim_{x\to a}\ \sqrt[n]{f(x)} \ = \ \sqrt[n]{\lim_{x\to a}\ f(x)}$ *provided* $n \ \in \ \mathbb{Z}^+$

These Limit Laws enable us to evaluate limits.

ONLY USE THESE LAWS IF THE DIRECTIONS SAY JUSTIFY EACH STEP

Otherwise there are shortcuts that we will get to in a minute.

ex 20 Find $\lim_{x\to 5}\ (2x^2-3x+4)$ by justifying each step.

$$\begin{aligned}\lim_{x\to 5}\ (2x^2-3x+4) \ &= \ \lim_{x\to 5}\ 2x^2 \ - \ \lim_{x\to 5} 3x \ + \ \lim_{x\to 5}\ 4 \ \ \text{(Rule 1 and Rule 2)}\\ &= \ 2\ \lim_{x\to 5}\ x^2 \ - \ 3\ \lim_{x\to 5}\ x \ + \ \lim_{x\to 5}\ 4 \ \ \text{(Rule 3)}\\ &= \ 2(25) \ - \ 3(5)+4 \ \ \text{(Rule 7, Rule 8 and Rule 9)}\\ &= \ 39 \ \ \text{(Rules from Grade School)}\end{aligned}$$

As you can clearly see this is a tedious and boring process. Again, you will only need to go about it this way if the directions say... justify each step!

OTHERWISE, proceed as follows. Given a limit, without justification in the directions of course, first try the ***Direct Substitution Property***, which says

If f is a polynomial OR a rational function AND a is in the domain of f, then

$$\lim_{x\ \to\ a}\ f(x) \ = \ f(a)$$

In other words, you can just plug it in. Provided it is in the domain of course!

ex 21 Find

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

Does Direct Substitution work here? Why not? Now what?

If Direct Substitution doesn't work then try the following strategies:

1. Factor
2. Expand
3. Rationalize
4. Manipulate Algebraically

Which of those will help here?

Since

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} \implies \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

BE CAREFUL!! Does

$$\frac{x^2-1}{x-1} = x+1 \; ? \qquad NO!$$

Then why is the previous method still correct??

ex 22 Find

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$$

Again, Direct Substitution will not work. Why? What will work?

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right) = \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{1}{(\sqrt{1+h}+1)} = \frac{1}{(\sqrt{1+0}+1)} = \frac{1}{2}$$

4.4 Limits at Infinity

Sometimes we would like to know what happens to a function as x gets very large

Consider the following limit:

$$\lim_{x\to\infty} \frac{4x^2 - 1}{x^2 + 2}$$

If you begin to plug in values of x, what happens?

x	$f(x)$
5	3.666
10	3.9117
100	3.999

It seems as if the fraction gets closer and closer to 4

Definition

We write:

$$\lim_{x\to\infty} f(x) = L$$

We say:

the limit of $f(x)$ as x approaches infinity is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently large

Also

We write:

$$\lim_{x\to-\infty} f(x) = L$$

We say:

the limit of $f(x)$ as x approaches negative infinity is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently large and negative

* Note that ∞ and $-\infty$ do NOT represent actual numbers *

ex 23 Investigate

$$\lim_{x\to\infty} \frac{1}{x}$$

What happens to the function as x gets very large? It looks like the limit is 0.

Theorem

For all $n > 0$,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

and

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

provided $\frac{1}{x^n}$ is defined.

We will use this fact to evaluate limits as x approaches ∞ OR $-\infty$

ex 24 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

The technique is as follows. Since we are only concerned with large values of x, divide EACH TERM by the highest power in the DENOMINATOR. Then we can utilize the previous theorem.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

Now, keeping in mind the theorem, what happens to the following as $x \longrightarrow \infty$?

$$\frac{1}{x}\ ,\ \frac{2}{x^2}\ ,\ \frac{4}{x}\ \ and\ \ \frac{1}{x^2}$$

They all go to 0! Thus,

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

ex 25 Find

$$\lim_{x\to\infty} \frac{4x^2-1}{5x+4}$$

$$\lim_{x\to\infty} \frac{4x^2-1}{5x+4} = \lim_{x\to\infty} \frac{\frac{4x^2}{x}-\frac{1}{x}}{\frac{5x}{x}+\frac{4}{x}} = \lim_{x\to\infty} \frac{4x-\frac{1}{x}}{5+\frac{4}{x}} = \lim_{x\to\infty} \frac{4x}{5} = \infty$$

ex 26 Evaluate

$$\lim_{x\to\infty} \frac{2x^2+x-5}{15x^4-x^2}$$

Try this one on your own

5 One Sided Limits and Continuity

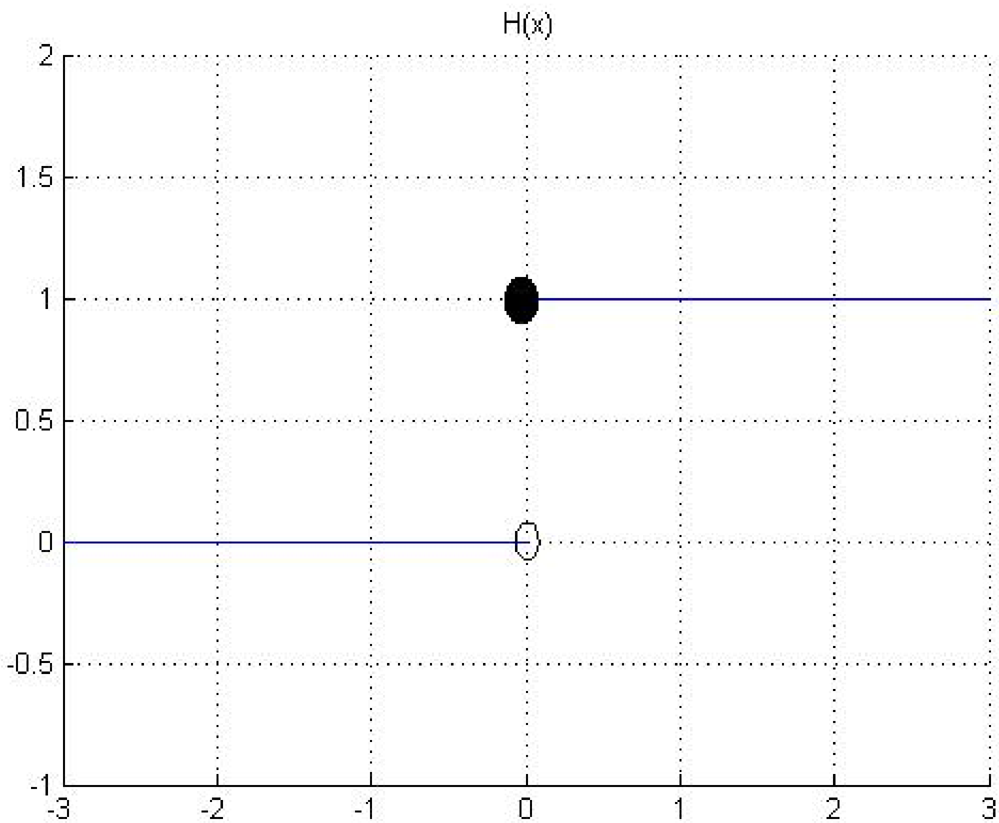
5.1 One Sided Limits

ex 27 Let

$$H(x) = \begin{cases} 1, & if\ x \geq 0 \\ 0, & if\ x < 0 \end{cases}$$

Find

$$\lim_{x \to 0} H(x)$$



Note that:

$$\lim_{x \to 0^-} H(x) = 0 \qquad (from\ the\ left)$$
$$\lim_{x \to 0^+} H(x) = 1 \qquad (from\ the\ right)$$

Therefore

$$\lim_{x \to 0} H(x) = DNE$$

Definition

We write:

$$\lim_{x\to a^-} \ f(x) \ = \ L$$

We say:

the limit of $f(x)$ as x approaches a from the left is L

We mean:

we can make the values of $f(x)$ as close to L as we want by
choosing values of x sufficiently close to **BUT LESS THAN** a

Also

We write:

$$\lim_{x\to a^+} \ f(x) \ = \ L$$

We say:

the limit of $f(x)$ as x approaches a from the right is L

We mean:

we can make the values of $f(x)$ as close to L as we want by
choosing values of x sufficiently close to **BUT GREATER THAN** a

Thus

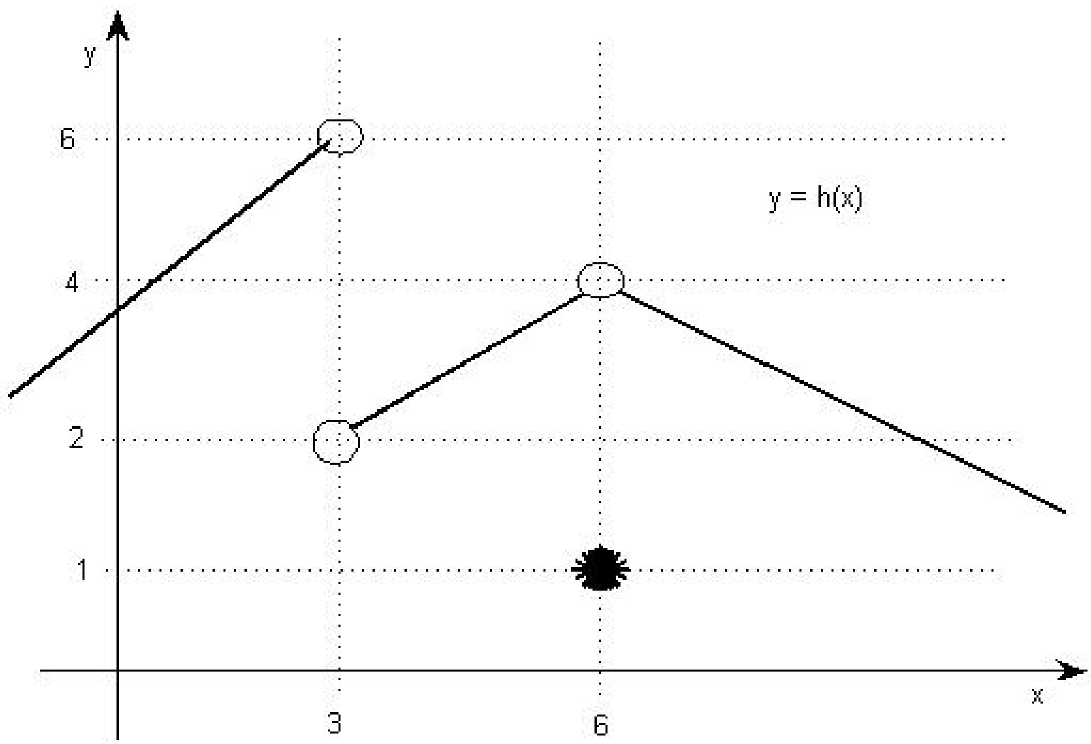
$$\lim_{x\to a} \ f(x) \ = \ L \quad \Longleftrightarrow \quad \lim_{x\to a^-} \ f(x) \ = \ L \ \ AND \ \ \lim_{x\to a^+} \ f(x) \ = \ L$$

ex 28 Find

$$\lim_{x \to 3^+} \frac{3+x}{3-x}$$

$$\lim_{x \to 3^+} \frac{3+x}{3-x} \;=\; \frac{6}{\textit{very small negative numbers}} \;=\; -\infty$$

ex 29 Find the following, if they exist:



$$\lim_{x \to 3^+} h(x) = \qquad \lim_{x \to 3^-} h(x) = \qquad \lim_{x \to 3} h(x) =$$

$$\lim_{x \to 6^+} h(x) = \qquad \lim_{x \to 3^-} h(x) = \qquad \lim_{x \to 6} h(x) =$$

$$\textit{how about } h(6) \;=\; ?$$

5.2 Continuity

Definition

A function is *continuous at a number a* if

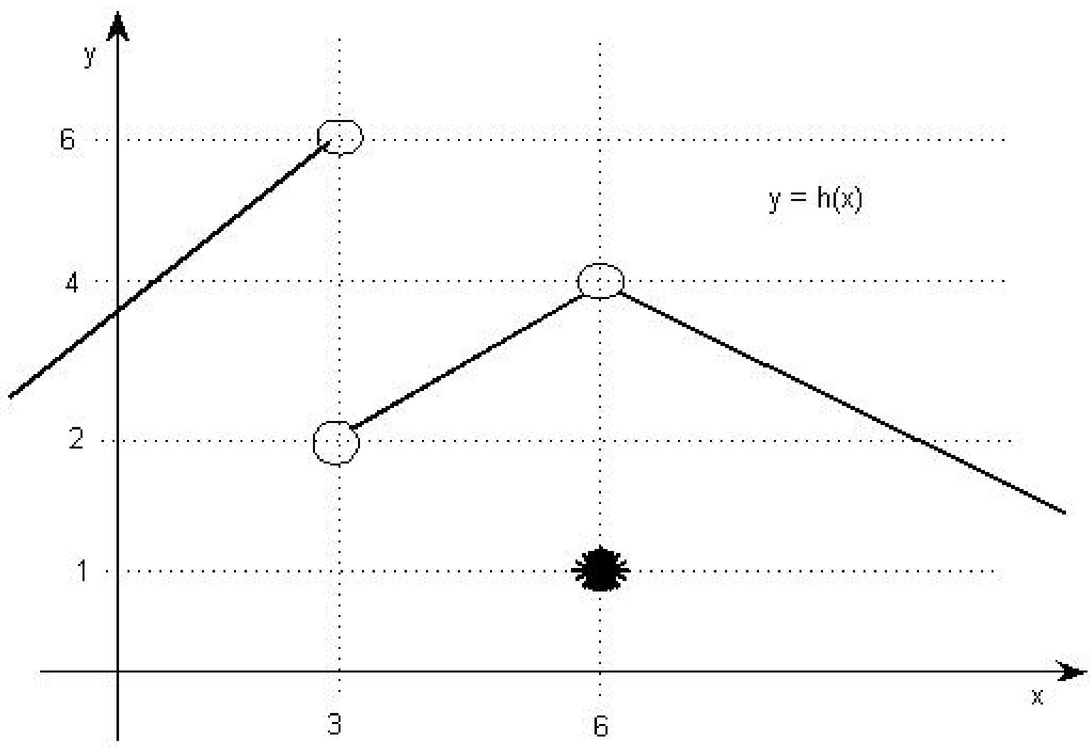
$$\lim_{x \to a} f(x) = f(a)$$

It is absolutely crucial to realize that this definition requires three (3) things:

1. $f(a)$ is defined, that is, a is in the domain of f
2. $\lim_{x \to a} f(x)$ exists, AND
3. $\lim_{x \to a} f(x) = f(a)$, that is, the numbers you get from (1) and (2) are THE SAME

If f is not continuous, it is called *discontinuous*

ex 30 Where is $h(x)$ discontinuous?



at $a = 3$, since both (1) and (2) are violated and at
$a = 6$ since (3) is violated $\quad h(6) = 1$ and $\lim_{x \to 6} h(x) = 4$ and clearly $1 \neq 4$

ex 31 Where is $f(x)$ discontinuous?

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

at $x = 2$ Why? 2 is NOT in the domain of $f(x)$

ex 32 Where is $f(x)$ discontinuous?

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Well now 2 is in the domain. In fact $f(2) = 1$ but $f(x)$ is still discontinuous at $x = 2$ Why?

What is $\lim_{x\to 2} f(x)$? Is that the same as $f(2)$?

Definition

A function f is *continuous on an interval* if it is continuous at every number on that interval.

Theorem

If f and g are continuous at a and c is a constant, then the following are also continuous at a

1. $f + g$
2. $f - g$
3. $c(f)$
4. fg
5. f/g, provided $g \neq 0$

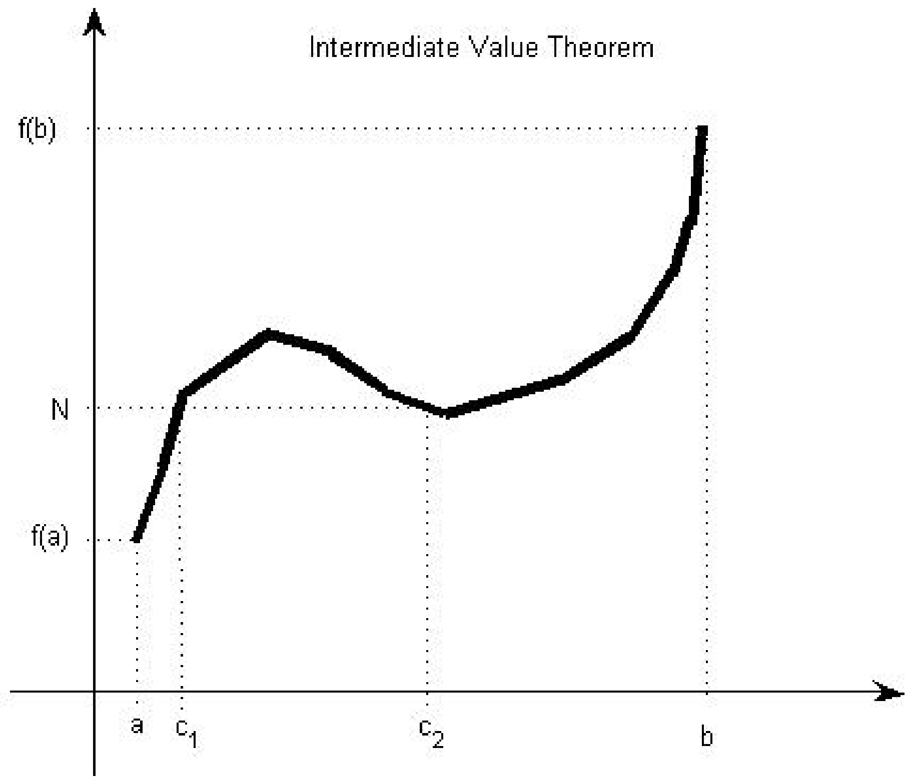
Theorem

Any polynomial is continuous on $\mathbb{R}$

Any Rational Function is continuous on its domain

The Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must exist a number c in (a, b) such that $f(c) = N$



Here there are 2 values where $f(c) = N$, c_1 and c_2

Why do you think this must happen? What is it about the function that prevents it from "hopping over" any values?

ex 33 How can we show that there is a root of $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2?

So, we are looking for a c between 1 and 2 such that $f(c) = 0$

But, $f(1) = -1 < 0$ and
$f(2) = 12 > 0$ and since f is continuous, by the IVT there must be at least one root between 1 and 2.

Why?

6 The Derivative

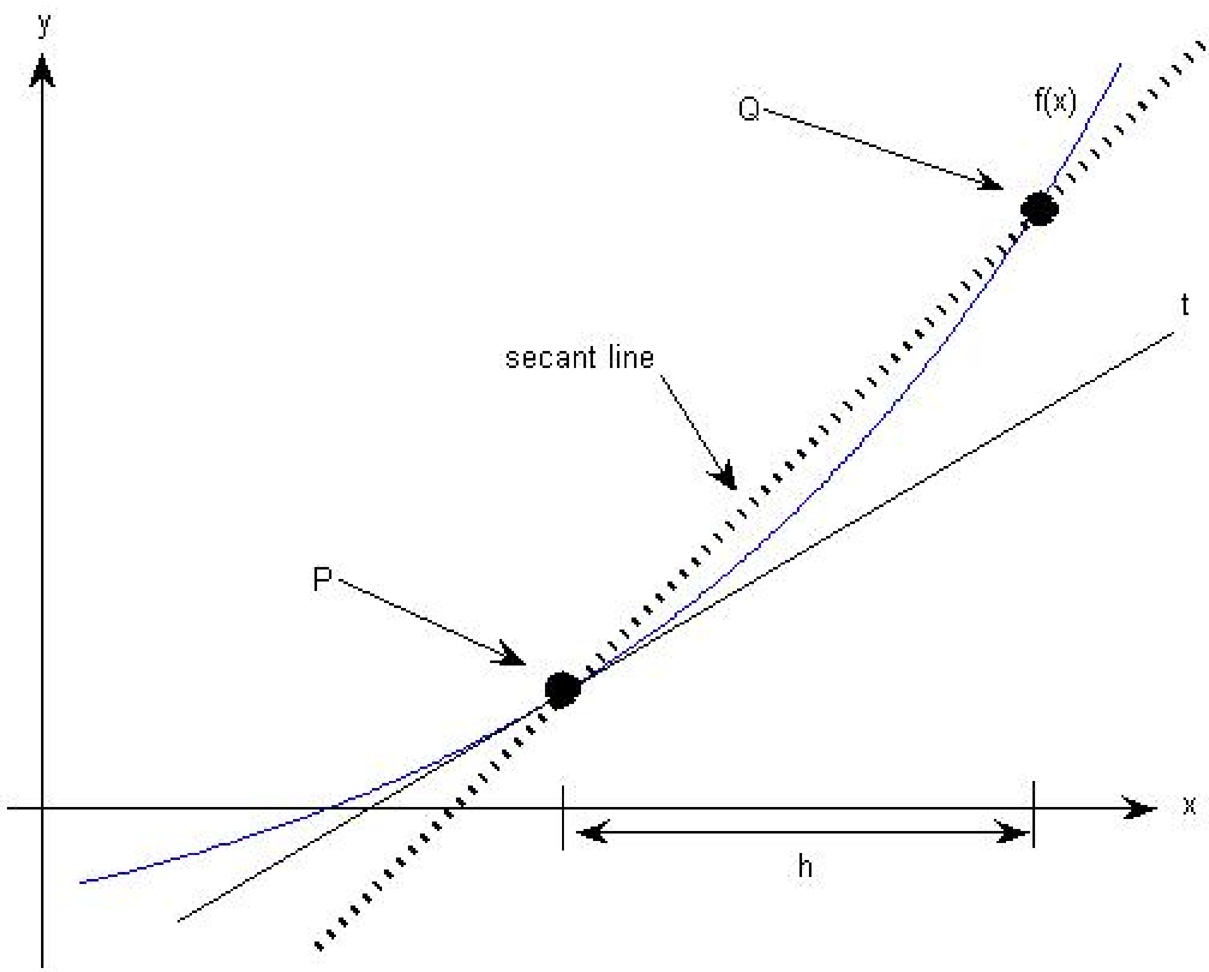
The first part of calculus deals with the derivative. This was developed as the solution to finding the slope of the tangent line.

Tangent basically means touching so a line tangent to a curve is a line that just touches the curve.

ex 34 Let's try to find an equation of the tangent line to the parabola $y = x^2$ at the point P

In order to make an equation of a line we need either two points or one point and the slope of the line. Since we only have one point, P, we will need to find the slope. The question is how does one find the slope of a line given only one point?

Let's choose a nearby point on the parabola, Q. Now, Q needs to be on the parabola, why? What are the coordinates of Q?



We can *approximate* the slope of the tangent line using the **secant line** connecting $P(x, f(x))$ and $Q(x+h, f(x+h))$

$$\textit{the slope is} \quad \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

What happens as Q gets closer to P, in other words as h approaches 0?

Now that we understand the concept of limits the *slope of the tangent line* to the function f at the point $P(x, f(x))$ is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Recall average velocity?

$$\textit{average velocity} \quad = \quad \frac{\textit{change in distance}}{\textit{change in time}}$$

What if the time, t, varies from $t = x$ to $t = x + h$ for some time interval h?

$$\textit{average velocity} \quad = \quad \frac{f(x+h) - f(x)}{(x+h) - x} \quad = \quad \frac{f(x+h) - f(x)}{h}^{*}$$

Look familiar?

The fraction * is called the *difference quotient* and it represents the AVERAGE RATE OF CHANGE

How do you think we represent the INSTANTANEOUS RATE OF CHANGE?

TAKE THE LIMIT

$$\textit{AVERAGE RATE OF CHANGE} \quad \longrightarrow \quad \frac{f(x+h) - f(x)}{h}$$

$$\textit{INSTANTANEOUS RATE OF CHANGE} \quad \longrightarrow \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The instantaneous rate of change, or as it's also known, the slope of the tangent line, has a special name in calculus. It is called the *derivative*.

Definition

The derivative of a function f , denoted $f'(x)$, is:

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Other notations are:

$$f'(x) \;=\; y' \;=\; \frac{dy}{dx}$$

ex 35 Find $f'(x)$ if $f(x) = 2x^2 - 3$

Let's use the definition, since for now that's all we have:

Since $f(x) = 2x^2 - 3$ we have

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h\to 0} \frac{2(x+h)^2 - 3 - (2x^2 - 3)}{h} \;=\; \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - 3 - 2x^2 + 3}{h}$$
$$=\; \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 3 - 2x^2 + 3}{h} \;=\; \lim_{h\to 0} \frac{4xh + 2h^2}{h} \;=\; \lim_{h\to 0} 4x + 2h \;=\; 4x$$

At each step you should quickly determine if you can use direct substitution. If not, continue until you can.

ex 36 Find the EQUATION of the tangent line to $y = 1/x$ at the point $(2, \frac{1}{2})$

Now in order to find the equation of ANY line you need either two points OR one point and the slope.

How fortunate for us that we now have a formula to determine the slope.

Again let's use the definition:

Since $f(x) = 1/x$ we have

$$m \;=\; f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \;=\; \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h} \;=\; \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h} \;=$$
$$\lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \;=\; \lim_{h\to 0} \frac{-h}{h(x)(x+h)} \;=\; \lim_{h\to 0} \frac{-1}{x(x+h)} \;=\; -\frac{1}{x^2} \;=\; -\frac{1}{4} \; \textit{when } x = 2$$

So the EQUATION is $y - \frac{1}{2} \;=\; -1/4(x - 2)$

Note that I have used point-slope form. You may use y-intercept if you insist but it will require some more work on your part to determine b

ex 37 Suppose that the distance a car travels is given by the function $f(t) = 2t^2 + 48t$. Find:

1. the average velocity from $[2, 4]$
2. the instantaneous velocity at $t = 2$

So, given that $t = 2$, $h = 2$ and $t + h = 4$ we have:

$$\textit{average velocity} = \frac{\Delta \textit{displacement}}{\Delta \textit{time}} = \frac{f(t+h) - f(t)}{h} = \frac{f(4) - f(2)}{2} = \frac{224 - 104}{2} = 60$$

Now at $t = 2$,

$$\textit{instantaneous velocity} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{2(t+h)^2 + 48(t+h) - (2t^2 + 48t)}{h}$$

$$= \lim_{h \to 0} \frac{2t^2 + 4th + 2h^2 + 48t + 48h - 2t^2 - 48t}{h} = \lim_{h \to 0} \frac{4th + 2h^2 + 48h}{h}$$

$$= \lim_{h \to 0} 4t + 2h + 48 = 4t + 48 \quad \textit{so at } t = 2 \textit{ the instantaneous velocity is } 56$$

ex 38 Is $y = \mid x \mid$ differentiable at $x = 0$?

Recall that:

$$\mid x \mid = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\mid x + h \mid - \mid x \mid}{h}$$

If

$$x \geq 0 \implies \mid x \mid = x \implies f'(x) = \lim_{h \to 0} \frac{(x+h) - (x)}{h} = 1$$

If

$$x < 0 \implies \mid x \mid = (-x) \implies f'(x) = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} = -1$$

$$\implies \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad DNE$$

Thus, $y = \mid x \mid$ is NOT differentiable.

Theorem

$$If\ a\ function\ is\ differentiable \quad \Longrightarrow \quad the\ function\ is\ continuous$$

Is the converse true? That is, does a continuous function have to be differentiable? NO! Can you think of a counterexample?

7 Rules of Derivatives

First, some notation. If I write:

$$\frac{d}{dx}(2x)$$

what I mean is, take the derivative of the function $2x$

So going back to example 35...

$$\frac{d}{dx}(2x^2 - 3) \;=\; 4x$$

Now the shortcuts...

If c is any constant:

$$\frac{d}{dx}(c) \;=\; 0$$

Why? Graphically what does $f(x) = c$ look like?

ex 39 If $f(x) = 28$, what is $f'(x)$

$$0$$

ex 40 If $f(x) = \pi$, what is $f'(x)$

$$0$$

The Power Rule

If n is any real number:

$$\frac{d}{dx}(x^n) \;=\; nx^{n-1}$$

ex 41 If $f(x) = x$, what is $f'(x)$

$$f(x) \;=\; x \;=\; x^1 \;\Longrightarrow\; f'(x) \;=\; (1)\; x^{(1-1)} \;=\; (1)x^0 \;=\; 1$$

ex 42 If $f(x) = x^6$, what is $f'(x)$

$$f(x) \;=\; x^6 \quad\Longrightarrow\quad f'(x) \;=\; (6)\; x^{(6-1)} \;=\; (6)x^5 \;=\; 6x^5$$

ex 43 If $f(x) = \frac{1}{\sqrt{x}}$, what is $f'(x)$

$$f(x) \;=\; \frac{1}{\sqrt{x}} \;=\; x^{-1/2} \quad\Longrightarrow\quad f'(x) \;=\; (-1/2)\; x^{(-1/2-1)} \;=\; (-1/2)x^{-3/2} \;=\; \frac{-1}{2\sqrt{x^3}}$$

ex 44 If $f(x) = x^{3/2}$, what is $f'(x)$

$$f(x) \;=\; x^{3/2} \quad\Longrightarrow\quad f'(x) \;=\; (3/2)\; x^{(3/2-1)} \;=\; (3/2)x^{1/2} \;=\; \frac{3\sqrt{x}}{2}$$

$$\frac{d}{dx}[c\; f(x)] \;=\; c\;\frac{d}{dx}[f(x)]$$

for any constant c

In other words, you can factor out constants.

ex 45 If $f(x) = 6x^2$, what is $f'(x)$

$$f(x) \;=\; 6x^2 \quad\Longrightarrow\quad f'(x) \;=\; (6)\;\frac{d}{dx}(x^2) \;=\; (6)\; 2x \;=\; 12x$$

ex 46 If $f(x) = \frac{12}{\sqrt[3]{x}}$, what is $f'(x)$

$$f(x) \;=\; \frac{12}{\sqrt[3]{x}} \;=\; 12\; x^{-1/3} \quad\Longrightarrow\quad f'(x) \;=\; 12\;\frac{d}{dx}(x^{-1/3}) \;=\; 12\;\left(-\frac{1}{3}\right) x^{-4/3} \;=\; -\,4x^{-4/3}$$

The Sum and Difference Rule

$$\frac{d}{dx}\;[f(x) \;\pm\; g(x)] \;=\; \frac{d}{dx}\;[f(x)] \;\pm\; \frac{d}{dx}\;[g(x)]$$

ex 47 If $f(x) = 3x^3 - x^2 + 4$, what is $f'(x)$

$$f(x) \;=\; 3x^3 - x^2 + 4 \;\Longrightarrow\; f'(x) \;=\; (3)\,\frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) + \frac{d}{dx}(4) \;=\; (3)\,3x^2 - 2x + 0 \;=\; 9x^2 - 2x$$

ex 48 If $f(x) = \frac{x^2}{6} + \frac{6}{x^3}$, what is $f'(x)$

$$f(x) \;=\; \frac{x^2}{6} + \frac{6}{x^3} \;=\; \left(\frac{1}{6}\right)\,x^2 + 6x^{-3} \;\Longrightarrow\; f'(x) \;=\; \frac{1}{6}\,\frac{d}{dx}(x^2) + 6\,\frac{d}{dx}(x^{-3})$$

$$=\; \frac{1}{6}\,(2x) + 6\,(-3)x^{-4} \;=\; \frac{x}{3} - \frac{18}{x^4}$$

ex 49 If the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$, where r is the radius in inches, find the rate of change in the volume when $r = 2/3$

This question is just another way to ask you to find the derivative of V and then plug in $r = 2/3$

Since π is a constant we have

$$V'(r) \;=\; \frac{4}{3}\,\pi\,\frac{d}{dr}(r^3) \;=\; \frac{4}{3}\,\pi\,3\,(r^2) \;=\; 4\,\pi\,r^2$$

When $r = 2/3$ we get

$$V'(r) \;=\; 4\,\pi\,(2/3)^2 \;=\; \frac{16\,\pi}{9}$$

8 Product and Quotient Rules

Although you can distribute the derivative through constants, subtraction and division it proves most unfortunate that it does not also work with multiplication and division. To show you this fact, observe:

Let $f(x) = x^2$ and $g(x) = x^3$

$$\frac{d}{dx}(f(x)g(x)) \;=\; \frac{d}{dx}((x^2)(x^3)) \;=\; \frac{d}{dx}(x^5) \;=\; 5x^4$$

However

$$\frac{d}{dx}(f(x))\,\frac{d}{dx}(g(x)) \;=\; \frac{d}{dx}(x^2)\,\frac{d}{dx}(x^3) \;=\; (2x)(3x^2) \;=\; 6x^3 \;\neq\; 5x^4$$

Therefore we require:

8.1 The Product Rule

$$\frac{d}{dx}[f(x)g(x)] \;=\; (f(x))\frac{d}{dx}(g(x)) \;+\; (g(x))\frac{d}{dx}(f(x))$$

Since the rule utilizes both multiplication and addition it is somewhat interchangeable. In other words, the derivative of a product of two functions is one multiplied by the derivative of the other added to the opposite. Order is not important because multiplication and addition are commutative. We will soon see that order DOES matter with the Quotient Rule.

ex 50 Find

$$\frac{d}{dx}[(x+1)(2x^2-3x+4)]$$

If we let $f(x) = x+1$ and $g(x) = 2x^2-3x+4$ then the product rule gives us

$$(f(x))\frac{d}{dx}(g(x)) \;+\; (g(x))\frac{d}{dx}(f(x)) \;=\; (x+1)\frac{d}{dx}(2x^2-3x+4) \;+\; (2x^2-3x+4)\frac{d}{dx}(x+1)$$

$$=\; (x+1)(4x-3)+(2x^2-3x+4)(1) \;=\; 4x^2+4x-3x-3+2x^2-3x+4 \;=\; 6x^2-2x+1$$

and YES you should simplify when its this simple

ex 51 Find

$$\frac{d}{dx}[(x^3 - 12x)(x^2 + 2x)]$$

If we let $f(x) = x^3 - 12x$ and $g(x) = x^2 + 2x$ then the product rule gives us

$$(f(x))\frac{d}{dx}(g(x)) \ + \ (g(x))\frac{d}{dx}(f(x)) \ = \ (x^3 - 12x)\frac{d}{dx}(x^2 + 2x) \ + \ (x^2 + 2x)\frac{d}{dx}(x^3 - 12x)$$

$$= \ (x^3 - 12x)(2x + 2) + (x^2 + 2x)(3x^2 - 12)$$

this one you can leave as is

8.2 The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \ = \ \frac{g(x)f'(x) - [f(x)g'(x)]}{[g(x)]^2}$$

Clearly this involves subtraction so you must pay attention to the order here.

If you ask me nicely I may prove this rule to you in class. It is actually an application of the Product Rule.

ex 52 Find

$$\frac{d}{dx}\left(\frac{x^2}{x+1}\right)$$

If we let $f(x) = x^2$ and $g(x) = x + 1$ then the quotient rule gives us

$$\frac{g(x)f'(x) - [f(x)g'(x)]}{[g(x)]^2} \ = \ \frac{(x+1)\frac{d}{dx}(x^2) - [(x^2)\frac{d}{dx}(x+1)]}{[x+1]^2}$$

$$= \ \frac{(x+1)(2x) - (x^2)(1)}{(x+1)^2} \ = \ \frac{2x^2 + 2x - x^2}{(x+1)^2} \ = \ \frac{x^2 + 2x}{(x+1)^2}$$

You will NEVER need to expand the denominator for me when using the quotient rule.

ex 53 Find

$$\frac{d}{dx}\left(\frac{x+\sqrt{x}}{3x-1}\right)$$

If we let $f(x) = x + \sqrt{x}$ and $g(x) = 3x - 1$ then the quotient rule gives us

$$\begin{aligned}\frac{g(x)f'(x) - [f(x)g'(x)]}{[g(x)]^2} &= \frac{(3x-1)\frac{d}{dx}(x+\sqrt{x}) - [(x+\sqrt{x})\frac{d}{dx}(3x-1)]}{[3x-1]^2} \\ &= \frac{(3x-1)(1+\frac{1}{2\sqrt{x}}) - (x+\sqrt{x})(3)}{(3x-1)^2}\end{aligned}$$

There is not much to simplify here so again you can leave it as is

9 The Chain Rule

If we let $h(x) = (x^3+x+1)^2$, finding $h'(x)$ would not be too difficult. You could always multiply it out and apply the previous rules.

What about $h(x) = (x^3 + x + 1)^{20}$? Would you enjoy multiplying that out? Probably not.

Technically this is what is known as a composite function, that is, of the form $(function\ in\ x)^{20}$

The chain rule shows us how to deal with taking the derivative of a composite function

9.1 The Chain Rule

If

$$h(x) = g[f(x)]$$

then

$$h'(x) = g'[f(x)]f'(x)$$

You can think of it as the derivative of the outside function times the derivative of the inside function

The General Power Rule

If

$$h(x) = [f(x)]^n$$

then

$$h'(x) = n[f(x)]^{n-1}f'(x)$$

OR

If

$$h(x) = [stuff]^n$$

then

$$h'(x) = n[stuff]^{n-1}\frac{d}{dx}(stuff)$$

ex 54 Find

$$\frac{d}{dx}\ (1-x)^3$$

So if $f(x) = 1 - x$ we have

$$n[f(x)]^{n-1}f'(x) \ = \ 3[1-x]^2\ \frac{d}{dx}\ (1-x) \ = \ 3(1-x)^2(-1) \ = \ -3(1-x)^2$$

ex 55 Find

$$\frac{d}{dx}\ 3(x^3-x)^4$$

If $f(x) = x^3 - x$ we have

$$n[f(x)]^{n-1}f'(x) \ = \ (3)(4)[x^3-x]^3\ \frac{d}{dx}\ (x^3-x) \ = \ 12(x^3-x)^3(3x^2-1)$$

ex 56 Find

$$\frac{d}{dt}\ \sqrt{12t^2+t}$$

Here we can rewrite the function as

$$(12t^2+t)^{1/2} \ \implies \ \frac{d}{dt} \ = \ \frac{1}{2}\ (12t^2+t)^{-1/2}\ \frac{d}{dt}(12t^2+t) \ = \ \frac{1}{2}\ (12t^2+t)^{-1/2}(24t+1)$$

ex 57 Find

$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right)^5$$

$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right)^5 \ = \ 5\ \left(\frac{x+1}{x-1}\right)^4\ \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$= \ 5\left(\frac{x+1}{x-1}\right)^4\left(\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}\right)$$

$$= \ 5\left(\frac{x+1}{x-1}\right)^4\left(\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right)$$

$$= \ 5\left(\frac{x+1}{x-1}\right)^4\left(\frac{-2}{(x-1)^2}\right) \ = \ -10\ \frac{(x+1)^4}{(x-1)^6}$$